B.Sc Physics Part-III
Paper-V, Group A
Mathematical Physics

Example on Harmonic Functions.

Q. Find the conjugate harmonic of $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$. Show that v is harmonic.

Soln.

Let $u + iv = f(z) = f(re^{i\theta})$, where u and v are expressed in terms of r and θ. The C-R equations are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{--- (1)}$$

and
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{--- (2)}$$

Since
$$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2 \quad \text{--- (3)}$$

$$\therefore \frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta \quad \text{--- (4)}$$

and
$$\frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \quad \text{--- (5)}$$

Now using eqs. (1) & (4), we obtain

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta - r\sin\theta \quad \text{--- (6)}$$

From eqs. (2) and (5), we get

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \quad \text{--- (7)}$$

From (6)
$$\frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta$$

Integrating above eq. with respect to r, we get

$$u = -r^2\sin 2\theta + r\sin\theta + \psi(\theta),$$

where $\psi(\theta)$ is an arbitrary function.

Now we can write

$$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta + \psi'(\theta) \quad \text{---} \quad (8)$$

From eqs. (7) and (8), we obtain

$$-2r^2\cos 2\theta + r\cos\theta = \frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta + \psi'(\theta)$$

which gives $\psi'(\theta) = 0$ or $\psi(\theta) = c$.

Therefore $$\boxed{u = -r^2\sin 2\theta + r\sin\theta + c} \quad \text{---} \quad (9)$$

Expression (9) shows the conjugate harmonic of v.

Next, v will be harmonic if it satisfies the condition

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

{ This is the Laplace eq. in polar coordinates (r, θ).

From eq. (4) we can write

$$\frac{\partial^2 v}{\partial \theta^2} = -4r^2\cos 2\theta + r\cos\theta$$

and from (5) we get,

$$\frac{\partial^2 v}{\partial r^2} = 2\cos 2\theta$$

Therefore, we obtain.

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

$$= 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta - \cos\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta + r\cos\theta)$$

$$= 4\cos 2\theta - \frac{1}{r}\cos\theta - 4\cos 2\theta + \frac{1}{r}\cos\theta$$

$$= 0$$

$\Rightarrow$ v is harmonic.